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Sums of slices in direct sums of Banach spaces

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Abstract. Motivated by recent studies on the strong diameter 2 property, we exhibit a simple reason why, in many direct sums of Banach spaces, the arithmetic mean of two slices may have diameter less than 2.

Key words: geometry of Banach spaces, direct sums, uniformly convex norm, slices, strong diameter 2 property.

Let *X* be a Banach space (over the real field \mathbb{R}). Let B_X and S_X denote its closed unit ball and its unit sphere, respectively. A *slice* of B_X is a set $S(x^*, \alpha) := \{x \in B_X : x^*(x) > 1 - \alpha\}$, where $x^* \in S_{X^*}$ and $\alpha > 0$.

The maximal diameter of slices can be 2. The same clearly holds for *convex combinations* of slices: diam $(\sum_{i=1}^n \lambda_i S_i) \le 2$ when S_i are slices and $\lambda_i \ge 0$ satisfy $\sum_{i=1}^n \lambda_i = 1$. In studies on "big" slice phenomena, some recent attention has been paid to Banach spaces X where the diameter of every convex combination of slices equals 2 (see, e.g., [1,2,4] for results and references). Such spaces are said to have the *strong diameter* 2 *property* (for this and related terminology, see [1]). For instance, it is known that the direct sums $c_0 \oplus_1 c_0$ and $c_0 \oplus_\infty c_0$ have the strong diameter 2 property.

Answering (in the negative) a question posed in [1, Question (c)], R. Haller, J. Langemets, and M. Põldvere showed that the ℓ_p -sum $c_0 \oplus_p c_0$ fails the strong diameter 2 property whenever $1 . More generally, they proved the theorem (see [4, Theorem 1]): if <math>X \neq \{0\}$ and $Y \neq \{0\}$ are Banach spaces and $1 , then <math>X \oplus_p Y$ fails the strong diameter 2 property.

The same result was independently obtained in [2, Theorem 3.2] in a more general context of absolute norms. In both cases, among others, two slices S_1 and S_2 were exhibited, whose *arithmetic mean* $1/2(S_1+S_2)$ had diameter strictly less than 2.

In this short note, we shall present a simple proof of the above result. Since we also would like to expose the *reason for the failure of the strong diameter* 2 *property* in $X \oplus_p Y$, we shall consider the following general case of $X \oplus Y$.

Definition. Let X and Y be Banach spaces. We say that $X \oplus Y$ is equipped with a uniformly convex \mathbb{R}^2 -norm if ||(x,y)|| = |(||x||,||y||)| for all $x \in X$ and $y \in Y$, where $|(\cdot,\cdot)|$ is a uniformly convex norm on \mathbb{R}^2 such that |(1,0)| = |(0,1)| = 1 and $|(a,b)| \le |(c,d)|$ whenever $0 \le a \le c$ and $0 \le b \le d$.

Recall (see, e.g., [5, pp. 59–60]) that a norm $\|\cdot\|$ of a Banach space X is *uniformly convex* if its *modulus of convexity* $\delta_X : (0,2] \to [0,1]$, defined by

$$\delta_{X}(\varepsilon) = \inf \left\{ 1 - \frac{\left\| x + y \right\|}{2} : x, y \in X, \left\| x \right\|, \left\| y \right\| \le 1, \left\| x - y \right\| \ge \varepsilon \right\},\,$$

is strictly positive (i.e., $\delta_X(\varepsilon) > 0$ when $\varepsilon > 0$).

The modulus of convexity measures (as nicely expressed in [3, p. 145]) the minimum depth of midpoints of line segments [x, y] in B_X below the unit sphere S_X . If X is uniformly convex, then S_X contains no (nontrivial) line segments, i.e., X is *strictly convex*. (The converse holds when X is of finite dimension.)

Let S be a non-empty subset of a Banach space $X = (X, \|\cdot\|)$. In the following theorem, we shall use the notation

$$||S|| := \sup \{||x|| : x \in S\}.$$

Since $S \subset ||S|| B_X$, we clearly have

$$\operatorname{diam} \frac{1}{2}S \leq \|S\|.$$

This is a rough estimate: if S is a slice, then ||S|| = 1, but diam S can be arbitrarily small (e.g., when $X = \ell_n^2, 1).$

Theorem. Let $X \neq \{0\}$ and $Y \neq \{0\}$ be Banach spaces. If $X \oplus Y$ is equipped with a uniformly convex \mathbb{R}^2 -norm, then there are slices S_1 and S_2 of $B_{X \oplus Y}$ such that

$$||S_1 + S_2|| < 2.$$

Consequently,

diam
$$\frac{1}{2}(S_1 + S_2) < 2$$
,

and $X \oplus Y$ fails the strong diameter 2 property.

Proof. Let $0 < \alpha < 1$. We choose α such that $(1 - \alpha, 1 - \alpha) \notin B_{\mathbb{R}^2}$. This can be done, because if $(1 - \alpha, 1 - \alpha) \in B_{\mathbb{R}^2}$ for all such α , then $(1, 1) \in B_{\mathbb{R}^2}$, meaning that $(1, 1) \in S_{\mathbb{R}^2}$. Hence, $S_{\mathbb{R}^2}$ would contain the line segment [(1, 0), (1, 1)], which is a contradiction.

Denote $\sigma_1 = \{(a,b) \in B_{\mathbb{R}^2} : a \ge 1 - \alpha\}$ and $\sigma_2 = \{(a,b) \in B_{\mathbb{R}^2} : b \ge 1 - \alpha\}$. Since σ_1 and σ_2 are disjoint non-empty compact subsets, one has $\varepsilon := \operatorname{dist}(\sigma_1, \sigma_2) > 0$ and $\delta := \delta_{\mathbb{R}^2}(\varepsilon) > 0$. If $u_i \in \sigma_i$, i = 1, 2, then $|u_1 - u_2| \ge \varepsilon$ and therefore $1 - |u_1 + u_2|/2 \ge \delta$. Hence $|u_1 + u_2| \le 2(1 - \delta)$, meaning that

$$|\sigma_1 + \sigma_2| \leq 2(1 - \delta)$$
.

Take now any $x^* \in S_{X^*}$ and $y^* \in S_{Y^*}$. Consider the slices $S_1 = S((x^*,0),\alpha)$ and $S_2 = S((0,y^*),\alpha)$ of $B_{X \oplus Y}$. We shall show that

$$||S_1 + S_2|| \le |\sigma_1 + \sigma_2|,$$

which would complete the proof of the theorem.

First, observe that if $(x,y) \in S_i$, then $(\|x\|, \|y\|) \in \sigma_i$, i = 1, 2. Indeed, $|(\|x\|, \|y\|)| = \|(x,y)\| \le 1$ and, e.g.,

$$||x|| \ge x^*(x) = (x^*, 0)(x, y) > 1 - \alpha.$$

Hence, for all $(x_i, y_i) \in S_i$, i = 1, 2, denoting $u_i = (||x_i||, ||y_i||) \in \sigma_i$, we have

$$||(x_1, y_1) + (x_2, y_2)|| = ||(x_1 + x_2, y_1 + y_2)||$$

$$= |(||x_1 + x_2||, ||y_1 + y_2||)|$$

$$\leq |(||x_1|| + ||x_2||, ||y_1|| + ||y_2||)|$$

$$= |u_1 + u_2|,$$

as desired.

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Viilude summad Banachi ruumide otsesummades

Eve Oia

On antud lihtne olemuslik põhjendus nähtusele, miks paljudes Banachi ruumide otsesummades võib viilude aritmeetilise keskmise diameeter olla väiksem kui 2. Tulemus on ajendatud hiljutistest uuringutest tugeva diameeter-2 omaduse vallas.