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NONLINEAR WAVES

Asymmetric waves in wave energy systems analysed by the stochastic Gauss–Lagrange wave model

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Abstract. The Gauss–Lagrange stochastic wave model is known to produce irregular waves with realistic degrees of asymmetry. We present the basic structure of the model and illustrate three of its characteristic properties: front–back asymmetry, particle orbits, and average horseshoe pattern. We also study the effect of a linear filter in a wave energy converting system on asymmetry and on average power of the system.

Key words: directional spreading, front–back asymmetry, horseshoe pattern, particle orbit, wave energy, wave steepness.

1. INTRODUCTION

Efficient design and control of wave energy converters (WEC) needs realistic and well parameterized descriptions of the wave environment, in which the converter shall operate. To catch the irregularity of real ocean waves, these models must contain stochastic elements, summarized in terms of statistical distributions. The Gaussian wave model is still, after more than fifty years [19], a standard model in naval architecture. It produces waves, which are statistically symmetric in the vertical and horizontal directions.

A physically motivated alternative is the Gauss–Lagrange model, or just Lagrange model, for the joint vertical and horizontal movements of individual water particles [4,6,18]. This model can produce irregular waves with realistic asymmetry properties. Initial studies of the statistical properties of Lagrange waves were made in 2006–2008 [1,2,9], dealing with crest-trough asymmetry. Detailed analyses of front–back asymmetry were first presented, for space waves in 2009 [14], for time waves in 2009–2010 [10,11], and for 3D waves in 2011 [13].

Stochastic wave models are commonly used as input in efficiency and reliability studies of ships and marine structures. In both situations it is the response of the system to the irregular seas that is of interest. For example, in a reliability analysis, extreme stress values and fatigue-causing load cycles are the result of a filtering process of the input wave loads, described in statistical terms. For efficient design, a construction has to behave in an optimal or nearly optimal way under many different conditions, all involving realistic wave irregularity and asymmetry. Wave asymmetry has different consequences in different situations. Fatigue damage is generally assumed to be unaffected of the exact time path of the varying load; only height and order of peaks and troughs are considered in most fatigue studies. In stability and impact studies of ships and marine installations, on the other hand, front–back wave asymmetry should be considered. In such cases, the Lagrange model offers a simple way to generate realistic waves in numerical simulation studies.

In this paper we will, in Section 2, briefly describe the Lagrange wave model. In Section 3 we illustrate by examples some statistical characteristics of the model, and in Section 4 we investigate some differences between the Gauss model and the Lagrange model when used as input to a WEC system in the form of a vertical point absorber. All computations are made in MATLAB and the toolbox WAFO [12,21].

2. THE LAGRANGE WAVE MODEL

The 2D stochastic Lagrange wave model is a stochastic version of Miche waves, the depth dependent modification of the Gerstner waves [5,15]. The Gerstner–Miche model describes the vertical and horizontal movements of individual water particle as functions of time t and original horizontal location. In the first order model, elementary components act independently of each other, and their effects are added. We consider here only particles on the free water surface. In the stochastic model the vertical and horizontal displacements are correlated random processes with time parameter t and space parameter u. The vertical process, denoted $W(t,u)$ is a Gaussian process and so is the horizontal process, denoted $X(t,u)$. Expressed verbally, in the stochastic 2D Lagrange wave model, a water particle with original still water location $(u,0)$ is, at time t, located at position

$$(X(t,u),W(t,u)). \tag{1}$$

The height of the water surface at location $x = X(t,u)$ is equal to $W(t,u)$. Due to the randomness in horizontal displacement, it may happen that more than one u-value satisfies $X(t,u) = x$; then the surface height is not uniquely defined. The probability of this *folding* is negligible, for all but very shallow waters.

The Lagrange wave model is defined as the pair $(X(t,u),W(t,u))$ of horizontal and vertical movement processes. When location is fixed, x_0, one obtains a *time wave* as a function of time t. Similarly, with time fixed, t_0, one obtains a *space wave* profile as a function of location x.

The Lagrange model is completely defined by the auto- and cross-covariance functions:

$$\begin{aligned} r^{ww}(t,u) &= \mathsf{Cov}(W(s,x),W(s+t,x+u)) \\ &= \int_0^\infty \cos(\kappa u - \omega t)\, S(\omega)\, \mathrm{d}\omega, \end{aligned}$$

and the analogues $r^{xx}(t,u), r^{wx}(t,u)$. Here, $S(\omega)$ is the *orbital spectrum*, and wave number $\kappa > 0$ and wave frequency $\omega > 0$ satisfy the depth dependent dispersion relation, $\omega^2 = g\kappa \tanh \kappa h$, with water depth h and gravitational constant g.

2.1. The Lagrange model with linked components

Expressed as stochastic Fourier integrals, the relation between the vertical and horizontal processes is

$$W(t,u) = \int_{-\infty}^{\infty} e^{i(\kappa u - \omega t)}\, \mathrm{d}\zeta(\omega), \tag{2}$$

$$X(t,u) = u + \int_{-\infty}^{\infty} e^{i(\kappa u - \omega t)}\, H(\omega)\, \mathrm{d}\zeta(\omega), \tag{3}$$

where the spectral process $\zeta(\omega)$ distributes the energy according to the orbital spectrum $S(\omega)$. In the simplest Lagrange model, the free, or Miche, model, the response function is $H(\omega) = i\frac{\cosh \kappa h}{\sinh \kappa h}$, which will give crest-trough asymmetric waves. A flexible general approach [14] is to let the response function be a general complex function, $H(\omega) = \rho(\omega)\, e^{i\theta(\omega)}$, leading to

$$\begin{aligned} r^{wx}(t,u) &= \mathsf{Cov}(W(s,x),X(s+t,x+u)) \\ &= \int_0^\infty \cos(\kappa u - \omega t + \theta(\omega))\, \rho(\omega)\, S(\omega)\, \mathrm{d}\omega, \end{aligned} \tag{4}$$

$$X(t,u) = u + \int_{-\infty}^{\infty} e^{i(\kappa u - \omega t + \theta(\omega))}\, \rho(\omega)\, \mathrm{d}\zeta(\omega). \tag{5}$$

We see that the free Lagrange model represents a phase shift between vertical and horizontal movement of $\theta = \pi/2 = 90^{\circ}$, while the general model has a frequency dependent phase shift.

In the free Lagrange model, individual water particles move unaffected by outer forces. The dependence between vertical and horizontal movements is taken care of by the Miche filtration. For wind driven waves this is unrealistic, and one would like to include external influence in the interaction. One way to formulate a relation between the vertical and horizontal processes is to let the horizontal acceleration of the water particles depend linearly on the vertical process, e.g., to take $X(t,u)$ as the solution to the equation

$$\frac{\partial^2 X(t,u)}{\partial t^2} = \frac{\partial^2 X_M(t,u)}{\partial t^2} - \alpha W(t,u), \tag{6}$$

with $\alpha > 0$. Here X_M is the Miche solution. With $G(\omega) = -\frac{\alpha}{(-i\omega)^2}$, the response function will be $H(\omega) = i\frac{\cosh \kappa h}{\sinh \kappa h} - \frac{\alpha}{(-i\omega)^2} = \rho(\omega)\, e^{i\theta(\omega)}$. By adjusting the values for the α-parameter one can obtain waves with realistic geometric properties, observed in empirical studies. Figure 1 shows an extreme example of asymmetrics plunging waves. For realistic wave spectra and water depth, this type of event occurs very rarely.

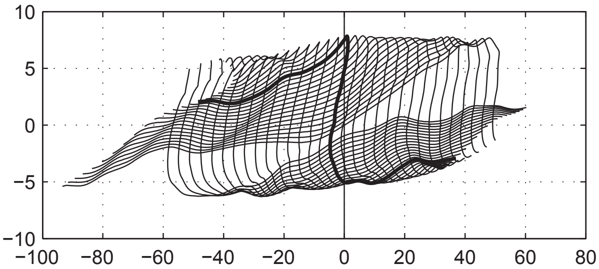

Fig. 1. Extreme asymmetric Lagrange waves with multiple points travelling from left to right. Time step is 0.25 seconds. The thick curve shows the wave profile at the instance when it is observed at the observation point at $x = 0$.

3. WAVE ASYMMETRY

3.1. Wave asymmetry measures

Many empirical studies have documented the degree of asymmetry in irregular waves, as measured with different indices. Figure 2 defines some wave characteristics used in these indices for a time wave. For analogous definitions for the space waves, the asymmetry is reversed with the steep side facing right.

In Section 3.2. we will show examples with similar degree of asymmetry as observed in experiments. We will report three different asymmetry measures. The first was defined in [14] as

$$\lambda_{AL} = -\frac{\mathsf{E}(L_t(t_{\text{down}}))}{\mathsf{E}(L_t(t_{\text{up}}))} \approx \frac{\mathsf{E}(H_{cb}/T_{cb})}{\mathsf{E}(H_{cf}/T_{cf})}, \tag{7}$$

where $L_t(t_{\text{down}})$ and $L_t(t_{\text{up}})$ are the slopes at down- and upcrossings of the mean water level. When front and back crest amplitudes are about the same, it is approximately equal to the index proposed in [17],

$$\lambda_{NLS} = \mathsf{E}(T_{cf})/\mathsf{E}(T_{cb}). \tag{8}$$

This index is related to $\lambda_{MK} = T'/T''$, proposed in [16]. The full distributions of T_{cb} and T_{cf} provide even more information about the asymmetry, and so do the T' and T'' distributions. Another measure is based on the Hilbert transform $\widehat{L}(t)$ and is defined from its third moment and standard deviation σ as

$$A = \mathsf{E}(\widehat{L}(t)^3)/\sigma^3. \tag{9}$$

This measure was used in [8] in a study of the relation between wind speed and wave asymmetry. The A-values reported in that work varied between zero and -0.4 for time waves, corresponding to steeper wave fronts than wave backs. In the examples we will stay within this range of asymmetry.

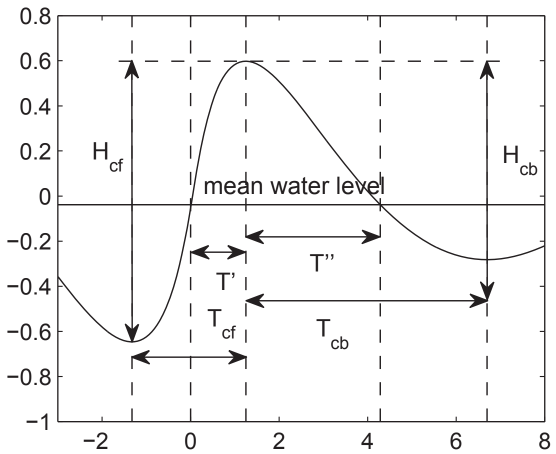

Fig. 2. Wave characteristic definitions for time waves.

3.2. Some characteristic properties of Lagrange wave asymmetry

Wave characteristic distributions can be found either by crossing theory (see [13] for examples and more references) or by Monte Carlo simulation, which is the method we chose here. The standard procedure to generate random waves is via a discrete approximation of the spectral integral and, for discrete wave numbers and frequencies, generate a discretized version of the integral (2),

$$W(t,u) = \sum A_k \cos(\kappa_k u - \omega_k t + \phi_k),$$

with phases ϕ_k uniformly distributed in $[0, 2\pi]$, and random or, most often, deterministic amplitudes A_k. We will use the discrete approximation with evenly spaced ω_k with spacing Δ_ω and corresponding wave numbers κ_k given by the dispersion relation. The amplitudes are random $A_k = \sqrt{\Delta_\omega S(\omega_k)}\sqrt{U_k^2 + V_k^2}$, with U_k, V_k independent standard normal variables. The horizontal process $X(t,u)$ is computed simultaneously from the corresponding formula. The computation is made by the fast Fourier transform in the time variable t, and looping over a discrete set of u-values. Finally, the Lagrange wave is computed according to the definition [12].

In our examples we will use an orbital spectrum $S(\omega)$ of the Pierson–Moskowitz (PM) type with significant wave height $H_s = 2.4$ m and different peak periods $T_p = 4, 6, 10$ s and truncated at 3 rad/s. The water depth is set to 20 m. (This combination of significant wave height and water depth is somewhat unrealistic, but is chosen to better illustrate the effect of linear filtering.)

Front–back asymmetry. From the simulations one can estimate the different asymmetry indices, which are shown in Table 1. Index λ_{AL} says that the rate of increase at mean level upcrossings is about twice the rate of decrease at downcrossings. Index λ_{NLS} says that the time from trough to crest is about $(70 \pm 5)\%$ of the time from crest to trough, on average. The A-indices agree with values reported in [8]. Cumulative slope distributions are shown in Fig. 3. The asymmetry parameter is $\alpha = 3$.

Particle orbits. In the Lagrange model with non-zero α-parameter the front-back asymmetry is related to the orientation of particle orbits, which will get a slightly upward tilt (Fig. 4).

Table 1. Front–back asymmetry measures (7–9) for Lagrange time waves with PM spectrum

T_p	4	6	10
λ_{AL}	0.57	0.50	0.42
λ_{NLS}	0.74	0.70	0.67
A	–0.32	–0.36	–0.44

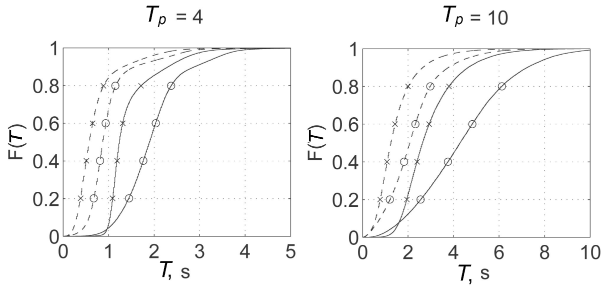

Fig. 3. CDF's of full crest front T_{tc}, T_{ct}, (x) and back (o) periods (solid) and corresponding half periods and T', T'' (dashed).

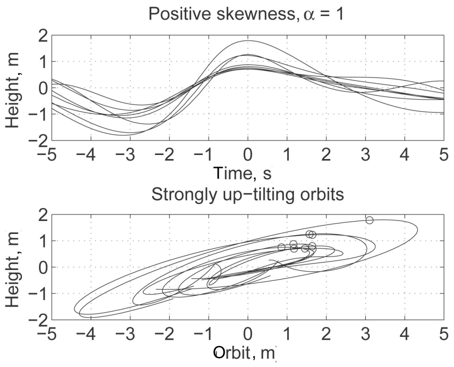

Fig. 4. Upper diagram shows eight positively skewed time waves (centered at time 0). The lower diagram shows the up-tilting orbits for the particles that are exactly at the crest in the upper diagram. Circles indicate their displacement at crest time.

Horseshoe pattern. Wave fields with directional spreading often exhibit crescent or horseshoe like wave patterns with concave wave back. The Lagrange models naturally produce such patterns, even with the first order model. Figure 5 shows the average shape of Lagrange waves near local maxima with specified height $u = 4$ m for PM directional orbital spectrum with standard spreading parameter $m = 5$. Asymmetry parameter is $\alpha = 2$.

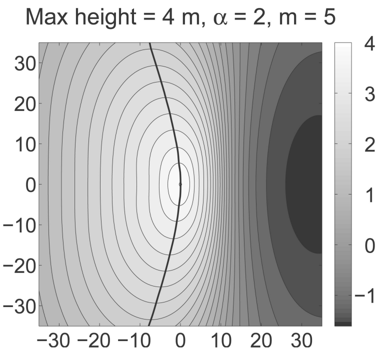

Fig. 5. Level curves for average 3D Lagrange wave height near local maximum with height $u = 4$ m, centered at the origin. Wave direction is from left to right; horizontal and vertical axis show distance from the maximum. Solid curve indicates the maximum of the average wave front.

4. ASYMMETRIC LAGRANGE WAVES IN A LINEAR WAVE ENERGY SYSTEM

Wave asymmetry may play a role in design and control of wave energy converters. We will investigate some effects of wave asymmetry for one common design, the vertical linear converter. This generator consists of a magnetic alternator anchored to the sea floor by a spring and attached to a buoy floating on the sea surface. When moving up and down with the waves, the magnet induces an electric current in the windings of a surrounding stator, also fixed to the see floor [3,20].

The average power P delivered by the generator is proportional to the square of its vertical velocity; if $Z(t)$ denotes the vertical position of the magnet relative its position at calm water, then $P = \gamma \mathsf{E}\left((dZ(t)/dt)^2\right)$, where γ is a damping coefficient, which can be changed according to wave conditions. The standard approach in design studies is to use either empirical data or artificial data generated by a Gaussian model. The question addressed here is to what extent, if any, the average power depends on the wave asymmetry as manifested in the Lagrange model as compared to the Gaussian model.

The simplest formulation of the vertical linear converter is the spring-and-damper filter with hydrostatic excitation only, disregarding any hydrodynamical forces. With L and Z denoting surface elevation and system elevation relative to the surface, the governing equation is

$$(m+m_a)Z'' + \gamma Z' + kZ = c(L-Z). \tag{10}$$

Here, $m+m_a$ is the total moving mass, including any added water mass (assumed to be frequency independent), γ is the total damping, equal to the damping coefficient in the generator plus any hydrodynamic damping, k is the spring constant in the anchor spring. The parameter c depends on the geometry and size of the buoy. For a circular buoy with radius r it is simply $c = \rho g 2\pi r^2$ in water with density ρ. This model only contains two parameters, the eigenfrequency $\omega_0 = \sqrt{\frac{k+c}{m+m_a}}$ of buoy/alternator, and the relative damping $\zeta = \frac{\gamma}{2\sqrt{(m+m_a)(k+c)}}$. The value of the parameter c is irrelevant for the comparison between the Gaussian and the Lagrangian wave model.

In irregular seas the wave excitation does not act with regular periodicity, since individual waves have variable amplitude and period, and the efficiency is measured as the statistical average P. We will now investigate how this measure depends on the assumed wave model – Gaussian or Lagrangian. The power of the linear generator has its maximum when its eigenperiod is equal to the wave period. In a random sea there is no fixed

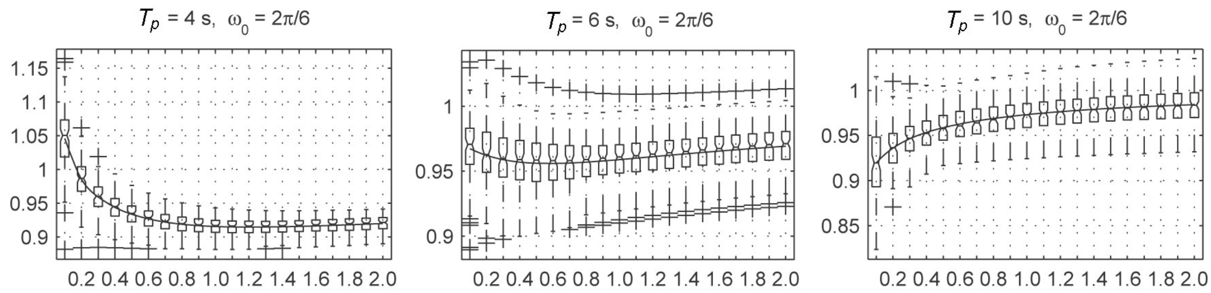

Fig. 6. Relative effect $P_{\mathrm{Lagrange}}/P_{\mathrm{Gauss}}$ as function of relative damping ζ for buoy/generator system with eigenfrequency $\omega_0 = 2\pi/6$ rad/s in three different seastates with peak period $T_p = 4, 6, 10$ s. The boxplots illustrate the spreading in 1000 simulations for each $\zeta = 0.1(0.1)2.0$.

wave period and one has to relate to the peak period of the power spectrum. Then there is the issue how much the wave asymmetry affects the theoretical average generator power.

Figure 6 shows boxplots of simulated ratios between the generator power when driven by front–back symmetric Gaussian waves and with asymmetric Lagrange waves. The orbital spectrum is a Pierson–Moskowitz spectrum with significant wave height 2.4 m and the water depth is 20 m. The ratio depends on the relative damping $\zeta = 0.1(0.1)2.0$ and on the wave peak period T_p and the eigenperiod ω_0. For each damping value 1000 wave sections were simulated, each 20 minutes long. The asymmetry parameter was $\alpha = 3$.

As seen in the figure, the use of a Gaussian wave model tends to mostly overestimate the generator power compared to what can be produced by asymmetric waves. However, as can be expected, the degree of front–back wave asymmetry is suppressed by the linear filter, and the vertical generator movement will be almost symmetric (not illustrated here). A nonlinear filter, taking also hydrodynamical forces into account, may give different results.

5. SUMMARY

The first order Gauss–Lagrange wave model consists of two linear wave components, which, when combined, produce waves that share many geometric properties typical for more complex nonlinear wave models. In the simplest Lagrange model there is a 90 deg phase shift between the vertical and horizontal components, giving crest-trough statistical asymmetry, depending on the water depth. A simple modification, governed by a single parameter, gives waves with realistic degrees of front–back asymmetry. A major advantage with the model is that one can compute the exact statistical distributions of different slope variables, without relying only on simulations.

The examples show what asymmetries that can be obtained, illustrate the statistical distributions of the crest front and crest back periods, and the tilted particle orbits near asymmetric wave crests. A new feature, not previously published, is the horseshoe like patterns near local crests in 3D Lagrange waves. The final example illustrates the consequences of using an asymmetric wave model in a linear wave energy converter, as compared to the often used Gaussian model.

A tutorial for how to simulate and analyse Lagrange models together with the WAFO toolbox is available in [12].

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Gaussi-Lagrange'i tüüpi juhuslike laineväljade asümmeetria ja energiasaagis

Georg Lindgren

Gaussi-Lagrange'i tüüpi juhuslike lainete mudel esitab kahest lineaarsest komponendist koosnevaid mitteregulaarseid lainevälju, mille asümmeetria omadused sarnanevad ookeanilainete vastavate omadustega. On esitatud sellise mudeli põhimõtteskeem ja arvutatud tekkivate veepinna kallete tõenäosusjaotused. On näidatud, milliseiks kujunevad lainete esi- ja taganõlva asümmeetria, veeosakeste trajektoorid ning laineharjade hobuserauakujuline muster ja demonstreeritud, et selliste lainete energiasaagis on väiksem kui sama kõrgete lineaarsete lainete puhul.